STABILITY RESULTS FOR IDENTIFYING AN UNKNOWN SOURCE TERM OF A HEAT EQUATION IN THE BANACH SPACE $L_1(\mathbb{R}^n)$

Nguyen Van Duc [(1)], **Luong Duy Nhat Minh** [(2)]
[1] *School of Natural Sciences Education, Vinh University, Vinh City, Nghe An Province*
[2] *Nam Ly 1 Secondary School, Dong Hoi City, Quang Binh Province*
Received on 03/01/2020, accepted for publication on 7/4/2020

Abstract: In this paper, we prove a stability estimate of Hölder type and propose a regularization method with error estimates of Hölder type for an inverse heat source problem in the Banach space $L_1(\mathbb{R}^n)$.
Keyword: Stability estimate; regularization method; inverse source problem.

1 Introduction

Inverse problems typically lead to mathematical models that are not well-posed in the sense of Hadamard [5]. This means especially that their solution is unstable under data perturbations. The two topics of concern in the field of inverse problems are the establishment of stability estimates ([1], [3], [8], [9], [10]) and the proposal of regularization methods ([2], [3], [6], [7], [9], [10]). In this paper, we establish stability estimates of Hölder type and propose a regularization method with error estimates of Hölder type for the n-dimensional inverse source problem of finding a pair of functions $\{u(x,t), f(x)\}$ with

$$u(\cdot,t) \in L_1(\mathbb{R}^n),\ \forall t \in [0,T], f(\cdot) \in L_1(\mathbb{R}^n)$$

satisfying:

$$\begin{cases} u_t = \Delta u + f(x)h(t),\ x \in \mathbb{R}^n, t \in (0,T) \\ u(x,0) = 0,\ x \in \mathbb{R}^n, \\ u(x,T) = g(x),\ x \in \mathbb{R}^n, \end{cases} \tag{1}$$

where Δ denotes the Laplace operator, g and h are given functions such that $g(\cdot) \in L_1(\mathbb{R}^n)$, $h : [0,T] \to \mathbb{R}$ is continuous on $[0,T]$, and $\int_0^T h(s)ds \neq 0$.

We would like to emphasize that although there are many results for inverse source problems of the heat equation in Hilbert spaces, there are very few results for this problem in Banach spaces, see, e.g., [1], [4], [11] and the references therein.

[1)] Email: nguyenvanducdhv@gmail.com (N. V. Duc)

With $q(\cdot) \in L_1(\mathbb{R}^n)$, we denote:

$$\begin{aligned} q^+ &:= \max\{q, 0\} \\ q^- &:= \max\{-q, 0\} \\ m(q) &:= \max\{\|q^+\|_1, \|q^-\|_1\} \\ n(q) &:= \min\{\|q^+\|_1, \|q^-\|_1\}. \end{aligned}$$

Remark 1.1. We have

$$\begin{aligned} q^+ &= (-q)^- \\ q^- &= (-q)^+. \end{aligned}$$

Definition 1.2. With any real number $k \geqslant 0$, we denote by S_k the following set of functions:

$$S_k = \{q : \mathbb{R}^n \to \mathbb{R}, q \in L_1(\mathbb{R}^n) \text{ such that } m(q) \geqslant (1 + k)n(q)\}. \tag{2}$$

Lemma 1.3. *The set S_k has the following properties:*
a) In case $k = 0$ then $S_0 \equiv L_1(\mathbb{R}^n)$,
b) $0 \in S_k,\ \forall k \geqslant 0$,
c) If $q \in L_1(\mathbb{R}^n)$ and q does not change sign over $\mathbb{R}^n$ then $q \in S_k$ for all $k \geqslant 0$,
d) If $q \in S_k$ then $-q \in S_k$,
e) If $q \in S_k$ then $\lambda q \in S_k,\ \forall \lambda \in \mathbb{R}$.

Proof. a) Clearly, for all $q \in L_1(\mathbb{R}^n)$ we have

$$\max\{\|q^+\|_1, \|q^-\|_1\} \geqslant \min\{\|q^+\|_1, \|q^-\|_1\}$$

or $m(q) \geqslant n(h) = (1 + 0)n(q)$. That means $q \in S_0$. Therefore $S_0 \equiv L_1(\mathbb{R}^n)$.
b) If $f = 0$ then $f^+ = f^- = 0$. Therefore $\|f^+\|_1 = \|f^-\|_1 = 0$. That implies

$$m(f) = n(f) = 0.$$

Hence the inequality $m(f) \geqslant (1 + k)n(f)$ holds true for all $k \geqslant 0$. So $0 \in S_k,\ \forall k \geqslant 0$.

c) If $q \in L_1(\mathbb{R}^n)$ and q does not change sign over $\mathbb{R}^n$ then $q^+ = \max\{q, 0\} = 0$ or

$$q^- = \max\{-q, 0\} = 0.$$

That deduces $\min\{\|q^+\|_1, \|q^-\|_1\} = 0$. Therefore for all $k \geqslant 0$ we have

$$m(q) = \max\{\|q^+\|_1, \|q^-\|_1\} \geqslant 0 = (1 + k)\min\{\|q^+\|_1, \|q^-\|_1\} = (1 + k)n(q).$$

So $q \in S_k$ for all $k \geqslant 0$.

d) Since $q^+ = (-q)^-, q^- = (-q)^+$ then we have

$$\begin{aligned} m(q) &= \max\{\|q^+\|_1, \|q^-\|_1\} \\ &= \max\{\|(-q)^-\|_1, \|(-q)^+\|_1\} = \max\{\|(-q)^+\|_1, \|(-q)^-\|_1\} = m(-q), \\ n(q) &= \min\{\|q^+\|_1, \|q^-\|_1\} \\ &= \min\{\|(-q)^-\|_1, \|(-q)^+\|_1\} = \min\{\|(-q)^+\|_1, \|(-q)^-\|_1\} = n(-q). \end{aligned}$$

If $q \in S_k$ then $q \in L_1(\mathbb{R}^n)$ and $m(q) \geqslant (1+k)n(q)$. So that we have $-q \in L_1(\mathbb{R}^n)$ and $m(-q) \geqslant (1+k)n(-q)$. That implies $-q \in S_k$.

e) We consider the following cases:
Case 1: If $\lambda = 0$ then $\lambda q = 0 \in S_k,\ \forall k \geqslant 0$ follow b).
Case 2: If $\lambda > 0$ then we have

$$\begin{aligned}(\lambda q)^+(x) &= \max\{(\lambda q)(x), 0\} = \max\{\lambda q(x), 0\} = \lambda \max\{q(x), 0\} = \lambda . q^+(x),\\ (\lambda q)^-(x) &= \max\{-(\lambda q)(x), 0\}\\ &= \max\{-\lambda q(x), 0\} = \lambda \max\{-q(x), 0\} = \lambda . q^-(x).\end{aligned}$$

We infer

$$\begin{aligned}\|(\lambda q)^+\|_1 = \|\lambda . q^+\|_1 = |\lambda| . \|q^+\|_1 = \lambda \|q^+\|_1,\\ \|(\lambda q)^-\|_1 = \|\lambda . q^-\|_1 = |\lambda| . \|q^-\|_1 = \lambda \|q^-\|_1.\end{aligned}$$

Then we have

$$\begin{aligned}m(\lambda q) &= \max\{\|(\lambda q)^+\|_1, \|(\lambda q)^-\|_1\} = \max\{\lambda\|q^+\|_1, \lambda\|q^-\|_1\}\\ &= \lambda \max\{\|q^+\|_1, \|q^-\|_1\} = \lambda m(q),\\ n(\lambda q) &= \min\{\|(\lambda q)^+\|_1, \|(\lambda q)^-\|_1\} = \min\{\lambda\|q^+\|_1, \lambda\|q^-\|_1\}\\ &= \lambda \min\{\|q^+\|_1, \|q^-\|_1\} = \lambda n(q).\end{aligned}$$

Because $q \in S_k$ then we have $m(q) \geqslant (1+k)n(q)$. Multiplying the two sides of this inequality by $\lambda > 0$ we have $\lambda m(q) \geqslant (1+k)\lambda n(q)$ or $m(\lambda q) \geqslant (1+k)n(\lambda q)$. So $\lambda q \in S_k$ for all $\lambda > 0$.
Case 3: If $\lambda < 0$ then $-\lambda > 0$. Because $q \in S_k$ then from *d*) we have $-q \in S_k$. The result of Case 2 showed that $(-\lambda).(-q) \in S_k$ or $\lambda q \in S_k$. □

2 Stability Estimates

We now provide a stability estimate for the solution of problem (1) on the class of function S_k with $k > 0$.

Theorem 2.1. *Suppose that $\{u_i, f_i\}, i = 1, 2$ where $f_1 - f_2 \in S_k$ are solutions of problem (1) corresponding to the data $g_i \in L_1(\mathbb{R}^n),\ i = 1, 2$, satisfying*

$$\|g_1 - g_2\|_1 \leqslant \delta, \tag{3}$$

then we have the following estimate

$$\|f_1 - f_2\|_1 \leqslant C\delta, \tag{4}$$

with $C = \dfrac{1}{\left|\int_0^T h(s)ds\right|}\left(1 + \dfrac{2}{k}\right)$.

Proof. We set

$$\begin{aligned}
&K(x,t) = \frac{1}{(2\sqrt{\pi t})^n} e^{-\frac{|x|^2}{4t}}, x \in \mathbb{R}^n, t > 0 \\
&f = f_1 - f_2 \\
&g = g_1 - g_2 \\
&u = u_1 - u_2.
\end{aligned}$$

Then $f \in S_k$, $g \in L_1(\mathbb{R}^n)$ and u, f, g satisfy problem (1), then we have (see [10])

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} K(x-y, t-s) f(y) h(s) dy ds. \tag{5}$$

Replacing t by T in (5) we have

$$g(x) = u(x,T) = \int_0^T \int_{\mathbb{R}^n} K(x-y, T-s) f(y) h(s) dy ds. \tag{6}$$

Integrating both sides of (6) and then changing the order of integration we have

$$\begin{aligned}
\int_{\mathbb{R}^n} g(x)dx &= \int_{\mathbb{R}^n} \left(\int_0^T \int_{\mathbb{R}^n} K(x-y, T-s) f(y) h(s) dy ds \right) dx \\
&= \int_{\mathbb{R}^n} \left(\left(\int_0^T \left(\int_{\mathbb{R}^n} K(x-y, T-s) dx \right) h(s) ds \right) f(y) \right) dy. \qquad (7)
\end{aligned}$$

Notice that $\int_{\mathbb{R}^n} K(x-y, T-s)dx = 1$, from (7) we have

$$\begin{aligned}
\int_{\mathbb{R}^n} g(x)dx &= \int_{\mathbb{R}^n} \left(\left(\int_0^T h(s)ds \right) f(y) \right) dy \\
&= \int_0^T h(s)ds. \int_{\mathbb{R}^n} f(y)dy = \int_0^T h(s)ds \int_{\mathbb{R}^n} f(x)dx.
\end{aligned}$$

Then we have

$$\int_{\mathbb{R}^n} f(x)dx = \frac{1}{\int_0^T h(s)ds} \int_{\mathbb{R}^n} g(x)dx. \tag{8}$$

Because $f \in S_k$ then we have $m(f) \geqslant (1+k)n(f)$ or

$$\begin{aligned}
&\max\{\|f^+\|_1, \|f^-\|_1\} \geqslant (1+k)\min\{\|f^+\|_1, \|f^-\|_1\} \\
\Leftrightarrow &\max\{\|f^+\|_1, \|f^-\|_1\} - \min\{\|f^+\|_1, \|f^-\|_1\} \geqslant k \min\{\|f^+\|_1, \|f^-\|_1\} \\
\Leftrightarrow &\min\{\|f^+\|_1, \|f^-\|_1\} \leqslant \frac{1}{k}\left(\max\{\|f^+\|_1, \|f^-\|_1\} - \min\{\|f^+\|_1, \|f^-\|_1\}\right) \\
\Leftrightarrow &\, 2\min\{\|f^+\|_1, \|f^-\|_1\} \leqslant \frac{2}{k}\left(\max\{\|f^+\|_1, \|f^-\|_1\} - \min\{\|f^+\|_1, \|f^-\|_1\}\right). \qquad (9)
\end{aligned}$$

Adding the two sides of the inequality (9) with the quantity $\max\{\|f^+\|_1, \|f^-\|_1\} - \min\{\|f^+\|_1, \|f^-\|_1\}$ we have

$$\begin{aligned}\max\{\|f^+\|_1, \|f^-\|_1\} &+ \min\{\|f^+\|_1, \|f^-\|_1\} \\ &\leqslant \left(1+\frac{2}{k}\right)\big(\max\{\|f^+\|_1, \|f^-\|_1\} - \min\{\|f^+\|_1, \|f^-\|_1\}\big)\end{aligned}$$

or

$$\|f^+\|_1 + \|f^-\|_1 \leqslant \left(1+\frac{2}{k}\right)\big(\max\{\|f^+\|_1, \|f^-\|_1\} - \min\{\|f^+\|_1, \|f^-\|_1\}\big). \tag{10}$$

On the other hand, we have

$$\begin{aligned}\|f\|_1 &= \int_{\mathbb{R}^n} |f(x)|dx = \int_{\mathbb{R}^n} (f^+(x) + f^-(x))dx \\ &= \int_{\mathbb{R}^n} f^+(x)dx + \int_{\mathbb{R}^n} f^-(x)dx \\ &= \int_{\mathbb{R}^n} |f^+(x)|dx + \int_{\mathbb{R}^n} |f^-(x)|dx \\ &= \|f^+\|_1 + \|f^-\|_1 \end{aligned} \tag{11}$$

and

$$\begin{aligned}\left|\int_{\mathbb{R}^n} f(x)dx\right| &= \left|\int_{\mathbb{R}^n} (f^+(x) - f^-(x))dx\right| \\ &= \left|\int_{\mathbb{R}^n} f^+(x)dx - \int_{\mathbb{R}^n} f^-(x))dx\right| \\ &= \left|\int_{\mathbb{R}^n} |f^+(x)|dx - \int_{\mathbb{R}^n} |f^-(x)|dx\right| \\ &= \big|\|f^+\|_1 - \|f^-\|_1\big| \\ &= \max\{\|f^+\|_1, \|f^-\|_1\} - \min\{\|f^+\|_1, \|f^-\|_1\}. \end{aligned} \tag{12}$$

From (8), (10) and (12) we have the estimate:

$$\begin{aligned}\|f\|_1 &\leqslant \left(1+\frac{2}{k}\right)\left|\int_{\mathbb{R}^n} f(x)dx\right| \\ &= \left(1+\frac{2}{k}\right)\left|\frac{1}{\int_0^T h(s)ds}\int_{\mathbb{R}^n} g(x)dx\right| \\ &\leqslant \frac{1}{\left|\int_0^T h(s)ds\right|}\left(1+\frac{2}{k}\right)\int_{\mathbb{R}^n} |g(x)|dx \\ &= \frac{1}{\left|\int_0^T h(s)ds\right|}\left(1+\frac{2}{k}\right)\|g\|_1 \\ &\leqslant C\delta. \end{aligned}$$

The theorem is proved. □

3 Regularization

We set $K_1(x,T) = \int_0^T K(x, T-s)h(s)ds,\ x \in \mathbb{R}^n$. From (6) we have

$$g = K_1 * f. \tag{13}$$

According to the proof of Theorem 2.1, if $f \in S_k$ then we have the following estimate:

$$\|f\|_1 \leqslant \frac{1}{\left|\int_0^T h(s)ds\right|}\left(1+\frac{2}{k}\right)\|g\|_1. \tag{14}$$

From (13) and (14) we have estimation

$$\|f\|_1 \leqslant \frac{1}{\left|\int_0^T h(s)ds\right|}\left(1+\frac{2}{k}\right)\|K_1 * f\|_1. \tag{15}$$

Then we have the lemma

Lemma 3.1. *If $q \in S_k$ with $k > 0$ then the following inequality holds:*

$$\|q\|_1 \leqslant C\|K_1 * q\|_1 \tag{16}$$

with $C = \dfrac{1}{\left|\int_0^T h(s)ds\right|}\left(1+\dfrac{2}{k}\right)$.

Now, with each $k > 0$, let M_k denote a subset of $L_1(\mathbb{R}^n)$ such that: if $q \in M_k$, $\widetilde{q} \in M_k$ then $q - \widetilde{q} \in S_k$.

Let us first take an example of a set satisfying the above property. For any functions p, q such that $p \in L_1(\mathbb{R}^n)$ and $q \in S_k$, we define the set M_k as follows

$$M_k = \{p + \alpha q : \alpha \in \mathbb{R}\}.$$

In this case, if $w \in M_k$, $\widetilde{w} \in M_k$ there exist real numbers α_1, α_2 so that

$$w = p + \alpha_1 q$$
$$\widetilde{w} = p + \alpha_2 q.$$

Hence we have $w - \widetilde{w} = (p + \alpha_1 q) - (p + \alpha_2 q) = (\alpha_1 - \alpha_2)q$. Since $q \in S_k$ then from Lemma 1.3, e) we have $(\alpha_1 - \alpha_2)q \in S_k$ or $w - \widetilde{w} \in S_k$.

Suppose that there exists a pair of functions $\{u, f\}$ with $u(\cdot, t) \in L_1(\mathbb{R}^n),\ \forall t \in [0,T], f \in M_k$ satisfying problem (1) in the case that function $g(\cdot) \in L_1(\mathbb{R}^n)$ is not known but we know an approximation $g^\delta(\cdot) \in L_1(\mathbb{R}^n)$ such that

$$\|g - g^\delta\|_1 \leqslant \delta. \tag{17}$$

The goal of the problem is to determine the function f from the noisy version g^δ of g.

We set $J(q) := \|K_1 * q - g^\delta\|^2, q \in M_k$. We obtain the following theorem:

Theorem 3.2. *Let $\tau > 0$ be a fixed real number. Choose $\bar{q} \in M_k$ such that*

$$J(\bar{q}) \leqslant \inf_{q \in M_k} J(h) + \tau\delta^2. \tag{18}$$

Then we have the following estimate

$$\|\bar{q} - f\|_1 \leqslant C_1\delta \tag{19}$$

with $C_1 = \dfrac{1}{\left|\int_0^T h(s)ds\right|}\left(1 + \dfrac{2}{k}\right)(1 + \sqrt{1+\tau})$.

Proof. From (18) we get

$$\begin{aligned}\|K_1 * \bar{q} - g^\delta\|_1^2 = J(\bar{q}) &\leqslant \inf_{q \in M_k} J(q) + \tau\delta^2 \\ &\leqslant J(f) + \tau\delta^2 \\ &= \|K_1 * f - g^\delta\|^2 + \tau\delta^2 \\ &= \|g - g^\delta\|^2 + \tau\delta^2 \\ &\leqslant \delta^2 + \tau\delta^2 = (1+\tau)\delta^2, \end{aligned} \tag{20}$$

From estimate (20) it implies that

$$\|K_1 * \bar{q} - g^\delta\|_1 \leqslant \sqrt{1+\tau}\delta. \tag{21}$$

From estimate (21) we have

$$\begin{aligned}\|K_1 * (\bar{q} - f)\|_1 &= \|K_1 * \bar{q} - K_1 * f\|_1 = \|K_1 * \bar{q} - g\|_1 \\ &= \|K_1 * \bar{q} - g^\delta + g^\delta - g\|_1 \\ &\leqslant \|K_1 * \bar{q} - g^\delta\|_1 + \|g^\delta - g\|_1 \\ &\leqslant \sqrt{1+\tau}\delta + \delta = (1 + \sqrt{1+\tau})\delta. \end{aligned} \tag{22}$$

Because $\bar{q} \in M_k$ and $f \in M_k$, we have $\bar{q} - f \in S_k$. Applying Lemma 3.1 we obtain

$$\|\bar{q} - f\|_1 \leqslant C\|K_1 * (\bar{q} - f)\|_1. \tag{23}$$

From (22) and (23) we conclude that

$$\|\bar{q} - f\|_1 \leqslant C(1 + \sqrt{1+\tau})\delta = C_1\delta.$$

The theorem is proved.

□

REFERENCES

[1] A. Ashyralyev, A. S. Erdogan and O. Demirdag, "On the determination of the right-hand side in a parabolic equation", *Applied Numerical Mathematics*, 62, pp. 1672–1683, 2012.

[2] N. V. Duc, "An a posteriori mollification method for the heat equation backward in time", *Journal of Inverse and Ill-Posed Problems*, 25(4), pp. 403-422, 2017.

[3] N. V. Duc, N. V. Thang, "Stability results for semi-linear parabolic equations backward in time", *Acta Mathematica Vietnamica*, 42(1), pp. 99–111, 2017.

[4] D. Guidetti, "Determining the Source Term in an Abstract Parabolic Problem From a Time Integral of the Solution", *Mediterranean Journal of Mathematics*, 9, pp. 611–633, 2012.

[5] J. Hadamard, *Lectures on the Cauchy Problem in Linear Partial Differential Equations*, Yale University Press, New Haven, 1923.

[6] D. N. Hao and N. V. Duc, "Regularization of backward parabolic equations in Banach spaces", *Journal of Inverse and Ill-Posed Problems*, 20, pp. 745-763, 2012.

[7] D. N. Hao and N. V. Duc, "A non-local boundary value problem method for semi-linear parabolic equations backward in time", *Applicable Analysis*, 94(3), pp. 446–463, 2015.

[8] D. N. Hao, N. V. Duc and N. V. Thang, "Stability estimates for Burgers-type equations backward in time", *Journal of Inverse and Ill-Posed Problems*, 23 (1), pp. 41–49, 2015.

[9] D. N. Hao, N. V. Duc and N. V. Thang, "Backward semi-linear parabolic equations with time-dependent coefficients and local Lipschitz source", *Inverse Problems*, 34(5), 0550109 (33pp), 2018.

[10] D. N. Hào, J. Liu, N. V. Duc and N. V. Thang, "Stability estimates backward time-fractional parabolic equations", *Inverse Problems*, 35, 125006 (25pp), 2019.

[11] A. Prilepko, S. Piskarev and S. Y. Shaw, "On approximation of inverse problem for abstract parabolic differential equations in Banach spaces", *Journal of Inverse and Ill-posed Problems*, 15(8), pp. 831–851, 2008.

[12] V. P. Mikhailov, *Partial Differential Equations*, Mir Publishers, 1978.

TÓM TẮT

CÁC KẾT QUẢ ỔN ĐỊNH CHO BÀI TOÁN XÁC ĐỊNH NGUỒN CỦA MỘT PHƯƠNG TRÌNH NHIỆT TRONG KHÔNG GIAN BANACH $L_1(\mathbb{R}^n)$

Trong bài báo này, chúng tôi chứng minh đánh giá ổn định kiểu Hölder và đề xuất một phương pháp chỉnh hóa với đánh giá sai số kiểu Hölder cho một bài toán nguồn nhiệt ngược trong không gian Banach $L_1(\mathbb{R}^n)$.

Từ khóa: Đánh giá ổn định; phương pháp chỉnh hóa; bài toán xác định nguồn.